Gauss - Seidel method

Consider the system of equations

$$a_1 x + b_1 y + c_1 z = d_1$$

$$a_2 x + b_2 y + c_2 z = d_2$$

$$a_3 x + b_3 y + c_3 z = d_3$$

Let us assume

$$|a_1| > |b_1| + |c_1|$$

$$|b_2| > |a_2| + |c_2|$$

$$|c_3| > |a_3| + |b_3|$$

ie, the coefficient of matrix should be diagonally dominant,

Solving the given system for x, y, z (whose coefficients are the larger values), we have

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y)$$

We start with the initial values $y = 0$, $z = 0$.

Condition for Convergence :

Gauss - Seidel method will converge if in each equation of the given system, the absolute value of the largest coefficient is greater than the sum of the absolute values of all the remaining coefficients.

ie, $$|a_{ii}| > \sum_{j=1}^{n} |a_{ij}| \quad \forall \ i = 1, 2 \ldots, n.$$

This is the sufficient condition for Convergence of Gauss- Seidel method.

PROBLEMS :

(1) Solve the following system of equations by Gauss - Seidel method.

$$27x + 6y - z = 85$$
$$x + y + 54z = 110$$
$$6x + 15y + 2z = 72.$$

Solution:

As the coefficient matrix is not diagonally dominant we rewrite the equations.

$$27x + 6y - z = 85$$

$$6x + 15y + 2z = 72$$

$$x + y + 54z = 110$$

Since the diagonal elements are dominant in the coefficient matrix, we write x, y, z as follows:

$$x = \frac{1}{27}[85 - 6y + z]$$

$$y = \frac{1}{15}[72 - 6x - 2z]$$

$$z = \frac{1}{54}[110 - x - y]$$

Let the initial values be $y = 0$, $z = 0$.

<u>First iteration</u>

$$x^{(1)} = \frac{1}{27}[85 - 6y + z] = \frac{1}{27}[85 - 6(0) + 0] = 3.148$$

$$y^{(1)} = \frac{1}{15}[72 - 6x - 2z] = \frac{1}{15}[72 - 6(3.148) - 0] = 3.541$$

$$z^{(1)} = \frac{1}{54}[110 - x - y] = \frac{1}{54}[110 - 3.148 - 3.541] = 1.913$$

Second iteration :

$$x^{(2)} = \frac{1}{27}[85-6y+z] = \frac{1}{27}[85-6(3.541)+1.913] = 2.432$$

$$y^{(2)} = \frac{1}{15}[72-6x-2z] = \frac{1}{15}[72-6(2.432)-2(1.913)] = 3.572$$

$$z^{(2)} = \frac{1}{54}[110-x-y] = \frac{1}{54}[110-2.432-3.572] = 1.926$$

Third iteration :

$$x^{(3)} = \frac{1}{27}[85-6y+z] = \frac{1}{27}[85-6(3.572)+1.926] = 2.426$$

$$y^{(3)} = \frac{1}{15}[72-6x-2z] = \frac{1}{15}[72-6(2.426)-2(1.926)] = 3.573$$

$$z^{(3)} = \frac{1}{54}[110-x-y] = \frac{1}{54}[110-2.426-3.573] = 1.926$$

Fourth iteration :

$$x^{(4)} = \frac{1}{27}[85-6y+z] = \frac{1}{27}[85-6(3.573)+1.926] = 2.426$$

$$y^{(4)} = \frac{1}{15}[72-6x-2z] = \frac{1}{15}[72-6(2.426)-2(1.926)] = 3.573$$

$$z^{(4)} = \frac{1}{54}[110-x-y] = \frac{1}{54}[110-2.426-3.573] = 1.926$$

Hence $\boxed{x = 2.426, \quad y = 3.573, \quad z = 1.926}$

(2) Solve the following system by Gauss - Seidal method.

$$28x + 4y - z = 32$$

$$x + 3y + 10z = 24$$

$$2x + 17y + 4z = 35$$

Solution :- Rewrite the equation as diagonally dominant.

$$28x + 4y - z = 32$$

$$2x + 17y + 4z = 35$$

$$x + 3y + 10z = 24$$

Let the initial values be $y=0,\ z=0$.

First iteration:

$$x^{(1)} = \frac{1}{28}[32 - 4y + z] = \frac{1}{28}[32 - 0 - 0] = 1.1429$$

$$y^{(1)} = \frac{1}{17}[35 - 2x - 4z] = \frac{1}{17}[35 - 2(1.1429) - 0] = 1.9244$$

$$z^{(1)} = \frac{1}{10}[24 - x - 3y] = \frac{1}{10}[24 - 1.1429 - 3(1.9244)] = 1.7084$$

Second iteration:

$$x^{(2)} = \frac{1}{28}[32 - 4y + z] = \frac{1}{28}[32 - 4(1.9244) + 1.7084]$$

$$= 0.9280$$

$$y^{(2)} = \frac{1}{17}[35 - 2x - 4z] = \frac{1}{17}[35 - 2(0.9230) - 4(1.7084)]$$

$$= 1.5483$$

$$z^{(2)} = \frac{1}{10}[24 - 0.923 - 3(1.5483)] = 1.8432.$$

<u>Third iteration:</u>

$$x^{(3)} = \frac{1}{28}[32 - 4(1.5483) + 1.8432] = 0.9875$$

$$y^{(3)} = \frac{1}{17}[35 - 2(0.9875) - 4(1.8432)] = 1.509$$

$$z^{(3)} = \frac{1}{10}[24 - 0.9875 - 3(1.509)] = 1.8486$$

<u>Fourth iteration:</u>

$$x^{(4)} = \frac{1}{28}[32 - 4(1.509) + 1.8486] = 0.9933$$

$$y^{(4)} = \frac{1}{17}[35 - (0.9933) - 4(1.8486)] = 1.507$$

$$z^{(4)} = \frac{1}{10}[24 - 0.9933 - 3(1.507)] = 1.8486$$

<u>Fifth iteration:</u>

$$x^{(5)} = \frac{1}{28}[32 - 4(1.507) + 1.8486] = 0.9936$$

$$y^{(5)} = \frac{1}{17}[35 - 2(0.9936) - 4(1.8486)] = 1.507$$

$$z^{(5)} = \frac{1}{10}[24 - 0.9986 - 3(1.507)] = 1.8486$$

Sixth iteration :

$$x^{(6)} = \frac{1}{28}[32 - 4(1.507) + 1.8486] = 0.9986$$

$$y^{(6)} = \frac{1}{17}[35 - 2(0.9986) - 4(1.8486)] = 1.507$$

$$z^{(6)} = \frac{1}{10}[24 - 0.9986 - 3(1.507)] = 1.8486$$

Here fifth iteration and sixth iteration are equal.

$\therefore$ Hence, the solution is

$$\boxed{\begin{aligned} x &= 0.9986 \\ y &= 1.507 \\ z &= 1.8486 \end{aligned}}$$

③. Solve the given system of equations by using Gauss - seidel iteration method.

$$20x + y - 2z = 17$$

$$3x + 20y - z = -18$$

$$2x - 3y + 20z = 25$$

Solution :-

As the coefficient matrix is diagonally dominant solving for x, y, z we get

$$x = \frac{1}{20}[17 - y + 2z]$$

$$y = \frac{1}{20}[-18 - 3x + z]$$

$$z = \frac{1}{20}[25 - 2x + 3y]$$

Let the initial values be $y = 0,\ z = 0$.

First iteration :

$$x^{(1)} = \frac{1}{20}[17 - 0 + 0] = \frac{17}{20} = 0.85$$

$$y^{(1)} = \frac{1}{20}[-18 - 3x + 0] = \frac{1}{20}[-18 - 2.55] = -1.0275$$

$$z^{(1)} = \frac{1}{20}[25 - 2x + 3y] = \frac{1}{20}[25 - 1.7 - 3.0825] = 1.0109$$

Second iteration :

$$x^{(2)} = \frac{1}{20}[17 - y + 2z] = \frac{1}{20}[17 + 1.0275 + 2.0218] = 1.0025$$

$$y^{(2)} = \frac{1}{20}[-18 - 3x + z] = \frac{1}{20}[-18 - 3.0075 + 1.0109] = -0.9998$$

$$z^{(2)} = \frac{1}{20}[25 - 2x + 3y] = \frac{1}{20}[25 - 2.005 - 2.9994] = 0.9998$$

Third iteration :

$$x^{(3)} = \frac{1}{20}[17 - y + 2z] = \frac{1}{20}[17 + 0.9998 + 1.9996] = 1$$

$$y^{(3)} = \frac{1}{20}[-18 - 3x + z] = \frac{1}{20}[-18 - 3 + 0.9998] = -1$$

$$z^{(3)} = \frac{1}{20}[25 - 2x + 3y] = \frac{1}{20}[25 - 2 - 3] = 1$$

Fourth iteration :

$$x^{(4)} = \frac{1}{20}[17 - y + 2z] = \frac{1}{20}[17 + 1 + 2] = 1$$

$$y^{(4)} = \frac{1}{20}[-18 - 3x + z] = \frac{1}{20}[-18 - 3 + 1] = -1$$

$$z^{(4)} = \frac{1}{20}[25 - 2x + 3y] = \frac{1}{20}[25 - 2 - 3] = 1 .$$

Hence $\boxed{x = 1,\ y = -1,\ z = 1}$

(Or)

Iteration	$x = \frac{1}{20}[17 - y + 2z]$	$y = \frac{1}{20}[-18 - 3x + z]$	$z = \frac{1}{20}[25 - 2x + 3y]$
1	0.85	-1.0275	1.0109
2	1.0025	-0.9998	0.9998
3	1	-1	1
4	1	-1	1

Hence $\boxed{x = 1,\ y = -1,\ z = 1.}$

Homework problems :

1. By using Gauss - Seidel method, solve the following system of equations

$6x + 3y + 12z = 35$, $8x - 3y + 2z = 20$, $4x + 11y - z = 33$.

Ans: $\boxed{x = 3.017 \,,\ y = 1.986 \,,\ z = 0.912}$.

2. Solve the following system of equations using Gauss - Seidel method.

$10x + 2y + z = 9$, $x + 10y - z = -22$, $-2x + 3y + 10z = 22$.

Ans: $\boxed{x = 1,\ y = -2,\ z = 3}$.

3. Solve by Gauss - Seidel method $3x + y = 2$, $x + 3y = -2$ correct to four decimal places.

Ans: $\boxed{x = 1 \,,\ y = -1}$.